

# KAIST POW 2012-9

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(Rank of a matrix) Let  $M$  be an  $n \times n$  matrix over the reals. Prove that  $\text{rank } M = \text{rank } M^2$  if and only if  $\lim_{\lambda \rightarrow 0} (\lambda I_n + M)^{-1} M$  exists.

**Proof.** Suppose that  $\text{rank } M = k$ . Then  $\exists \{B_1, \dots, B_k\} \subset \mathbb{R}^n$  a basis of the row  $M$ . Let  $B \in \mathbb{R}^{k \times n}$  with  $B_1, \dots, B_k$  as rows. Then  $M = AB$  for some  $A \in \mathbb{R}^{n \times k}$  since  $\text{row } M = \text{row } B$ . Then  $\text{col } A = \text{col } M$ . So we get  $\text{rank } A = \text{rank } B = k$ .

Remark that  $\exists \lim_{\lambda \rightarrow 0} X \Rightarrow \exists \lim_{\lambda \rightarrow 0} YX \wedge \exists \lim_{\lambda \rightarrow 0} XY$  for all constant matrix  $Y$  since every entry of  $XY$  and  $YX$  is a linear combination of some entries of  $X$ .

**Step1.**  $\text{rank } M^2 = \text{rank } M \Leftrightarrow BA$  is invertible.

Note that  $M^2 = (AB)(AB) = A(BA)B$ .  $BA$  is invertible  $\Leftrightarrow \text{ran}(BA)B = \text{ran } B$ , then  $\text{ran } M^2 = \text{ran } AB = \text{ran } M$ .

**Step2.**  $\exists \lim_{\lambda \rightarrow 0} (\lambda I_n + M)^{-1} M \Leftrightarrow \exists \lim_{\lambda \rightarrow 0} (\lambda I_k + BA)^{-1}$

Now we use the formula

$$(\lambda I_n + AB)^{-1} = \lambda^{-1} [I_n - A(\lambda I_k + BA)^{-1} B]$$

This is easily verified by matrix multiplication. So

$$(\lambda I_n + M)^{-1} M = (\lambda I_n + AB)^{-1} AB = I_n - \lambda(\lambda I_n + AB)^{-1} = A(\lambda I_k + BA)^{-1} B$$

Since  $\text{rank } A = \text{rank } B = k$ ,  $\text{col } B = \mathbb{R}^n$ . That is,  $\exists B' \in \mathbb{R}^{n \times k}$  such that  $BB' = I_k$ . By similar argument on row  $A$ ,  $\exists A' \in \mathbb{R}^{k \times n}$  such that  $A'A = I_k$ . So  $(\lambda I_k + BA)^{-1} = A'A(\lambda I_k + BA)^{-1} BB'$

Therefore,  $\exists \lim_{\lambda \rightarrow 0} A(\lambda I_k + BA)^{-1} B \Leftrightarrow \exists \lim_{\lambda \rightarrow 0} A'A(\lambda I_k + BA)^{-1} BB' = \lim_{\lambda \rightarrow 0} (\lambda I_k + BA)^{-1}$

**Step3.**  $\exists \lim_{\lambda \rightarrow 0} (\lambda I_k + BA)^{-1} \Leftrightarrow BA$  is invertible.

Note that  $(\lambda I_k + BA)^{-1} = \frac{1}{\det(\lambda I_k + BA)} \text{adj}(\lambda I_k + BA)$ .  $\lim_{\lambda \rightarrow 0} \text{adj}(\lambda I_k + BA) = \text{adj } BA$  and  $\lim_{\lambda \rightarrow 0} \det(\lambda I_k + BA) = \det BA$  because determinant and entries of the adjoint matrix are composed of product and sum of some entries from original matrix. So,  $\exists \lim_{\lambda \rightarrow 0} (\lambda I_k + BA)^{-1} \Leftrightarrow \det BA \neq 0 \Leftrightarrow BA$  is invertible.

Consequently,  $\text{rank } M = \text{rank } M^2 \Leftrightarrow \exists \lim_{\lambda \rightarrow 0} (\lambda I_n + M)^{-1} M$